۸۷/۲/۲۲ باسمه تعالی

فصل ۹ : مرکز ثقل (وزن) گرانی و مرکز هندسی

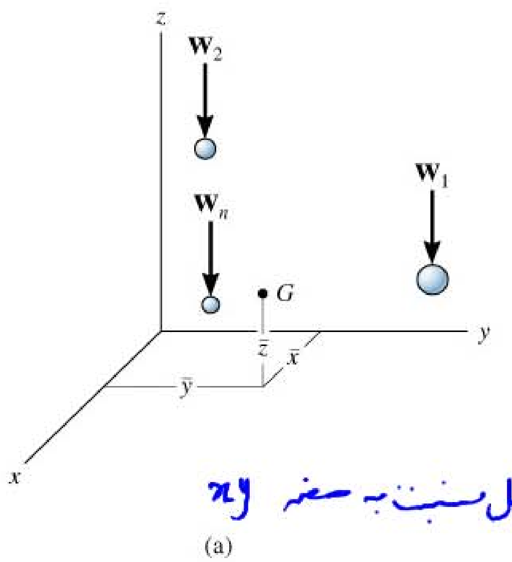
مرکز جرم منحنی 2D و 3D

سطح

حجم

مرکز ثقل $G \begin{cases} \bar{x} \\ \bar{y} \\ \bar{z} \end{cases}$

مرکز ثقل جسم :



(a)

با تبدیل نسبت به صفحه xy

$$W_R = \sum_{i=1}^{n} W_i$$

y,s: $W_R \bar{x} = \sum W_i \tilde{x}_i \longrightarrow \bar{x} = \frac{\sum W_i \tilde{x}_i}{\sum W_i}$

x,s: $W_R \bar{y} = \sum W_i \tilde{y}_i \longrightarrow \bar{y} = \frac{\sum W_i \tilde{y}_i}{\sum W_i}$

$W_R \bar{z} = \sum W_i \tilde{z}_i \longrightarrow \bar{z} = \frac{\sum W_i \tilde{z}_i}{\sum W_i}$

$\gamma = cte$

$$\bar{x} = \frac{\int \tilde{x}\, dw}{\int dw} = \frac{\int \tilde{x} \gamma\, dv}{\int \gamma\, dv} \overset{\text{همگن}}{=} \frac{\int \tilde{x}\, dv}{\int dv}$$ مرکز حجم

$$\bar{y} = \frac{\int \tilde{y}\, dw}{\int dw}$$

$$\bar{z} = \frac{\int \tilde{z}\, dw}{\int dw}$$

$g = cte$

$$\bar{x} = \frac{\int \tilde{x}\, dw}{\int dw} = \frac{\int \tilde{x} \rho g\, dv}{\int \rho g\, dv} = \frac{\int \tilde{x}\, dm}{\int dm}$$ مرکز جرم

2D و 3D

مرکز منحنی — مرکز سطح — مرکز حجم — مرکز جرم — مرکز ثقل

$$\bar{x} = \frac{\int \tilde{x}\, dL}{\int dL} \qquad \bar{x} = \frac{\int \tilde{x}\, dA}{\int dA} \qquad \bar{x} = \frac{\int \tilde{x}\, dV}{\int dV} \qquad \bar{x} = \frac{\int \tilde{x}\, dm}{\int dm} \qquad \bar{x} = \frac{\int \tilde{x}\, dW}{\int dW}$$

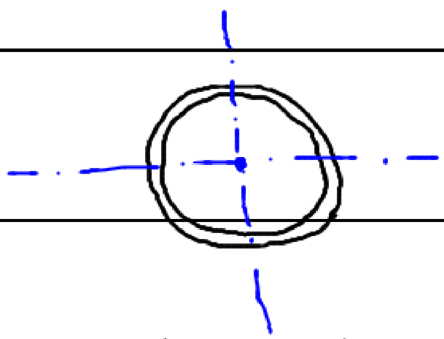
گشتاور اول جزء حجم نسبت به محور y $\quad \int \tilde{x}\, dV$

✓ مختصات قطبی یا کارتزین 2D

مختصات کروی، استوانه‌ای و کارتزین 3D

✓ اگر محور تقارن داشته باشد، مرکز هندسی (جسم) در امتداد محور تقارن است

✓ برای اجسام همگن، مرکز ثقل همان مرکز هندسی (جرم) است



روش محاسبه:

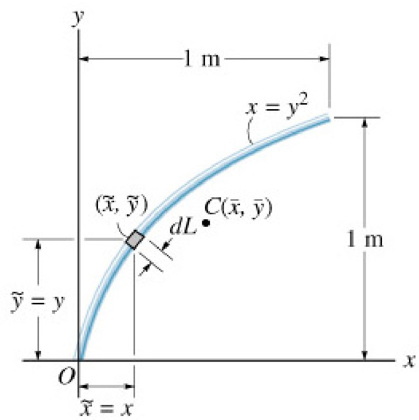
✓ انتخاب سیستم مختصات مناسب و جزء المان مناسب dV، dA، dL

مستطیل با طول محدود و جزء عرض

دیسک دایروی با شعاع محدود و جزء ضخامت

x, y, z ✓

انتگرال گیری ✓



مثال: مرکز منحنی روبرو را بدست آورید $\bar{x}, \bar{y}$

$$\bar{x} = \frac{\int \tilde{x}\, dL}{\int dL} \qquad dx = 2y\,dy$$

$$(dL)^2 = (dx)^2 + (dy)^2 \longrightarrow \left(\frac{dL}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1$$

$$dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\; dy$$

$$\bar{x} = \frac{\int x \sqrt{1+4y^2}\, dy}{\int \sqrt{1+4y^2}\, dy} = \frac{\int_0^1 y^2 \sqrt{1+4y^2}\, dy}{\int_0^1 \sqrt{1+4y^2}\, dy} =$$

$$\int \sqrt{x^2+a^2}\, dx = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2 \ln\left(x + \sqrt{x^2+a^2}\right)\right] + C$$

$$\int_0^1 \sqrt{1+4y^2}\, dy = 2\int_0^1 \sqrt{\frac{1}{4} + y^2}\; dy = \qquad a = \frac{1}{2}$$

$$2 \times \frac{1}{2}\left[y\sqrt{y^2 + \frac{1}{4}} + \frac{1}{4}\ln\left(y + \sqrt{y^2 + \frac{1}{4}}\right)\right]\Bigg|_0^1$$

$$\sqrt{\frac{5}{4}} + \frac{1}{4}\ln\left(1 + \sqrt{\frac{5}{4}}\right) - \frac{1}{4}\ln\frac{1}{2} = 1.479$$

$$\int x^2\sqrt{x^2+a^2}\, dx = \frac{x}{4}\sqrt{(x^2+a^2)^3} - \frac{a^2}{8}x\sqrt{x^2+a^2} - \frac{a^4}{8}\ln\left(x + \sqrt{x^2+a^2}\right)$$

$$\int_0^1 y^2\sqrt{1+4y^2}\,dy = 2\int_0^1 y^2\sqrt{\tfrac{1}{4}+y^2}\,dy \qquad a=\tfrac{1}{2}$$

$$2\left[\frac{y}{4}\sqrt{\left(y^2+\tfrac{1}{4}\right)^3} - \frac{1}{32}y\sqrt{y^2+\tfrac{1}{4}} - \frac{1}{128}\ln\left(y+\sqrt{y^2+\tfrac{1}{4}}\right)\right]_0^1$$

$$2\left[\frac{1}{4}\left(\frac{5}{4}\right)^{\frac{3}{2}} - \frac{1}{32}\left(\frac{5}{4}\right)^{\frac{1}{2}} - \frac{1}{128}\ln\left(1+\sqrt{\tfrac{5}{4}}\right) + \frac{1}{128}\ln\left(\tfrac{1}{2}\right)\right] = .6063$$

$$\bar{x} = \frac{.6063}{1.479} = .41 \text{ m}$$

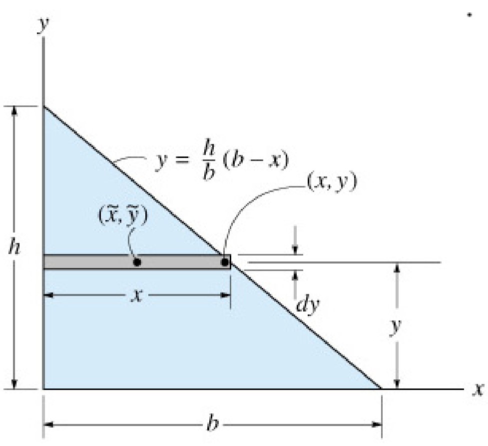
$$\bar{y} = \frac{\int \tilde{y}\,dL}{\int dL} = \frac{\int y\sqrt{1+4y^2}\,dy}{1.479}$$

$$\int x\sqrt{x^2+a^2}\,dx = \frac{1}{3}\sqrt{(x^2+a^2)^3}$$

$$2\int y\sqrt{\tfrac{1}{4}+y^2}\,dy = \frac{2}{3}\sqrt{\left(y^2+\tfrac{1}{4}\right)^3}\,\Bigg|_0^1 \qquad a=\tfrac{1}{2}$$

$$= \frac{2}{3}\left(\left(\frac{5}{4}\right)^{\frac{3}{2}} - \frac{1}{8}\right) = .8484$$

$$\bar{y} = \frac{.8484}{1.479} = .574 \text{ m}$$



مثال: $\bar{y} = ?$ مرکز سطح

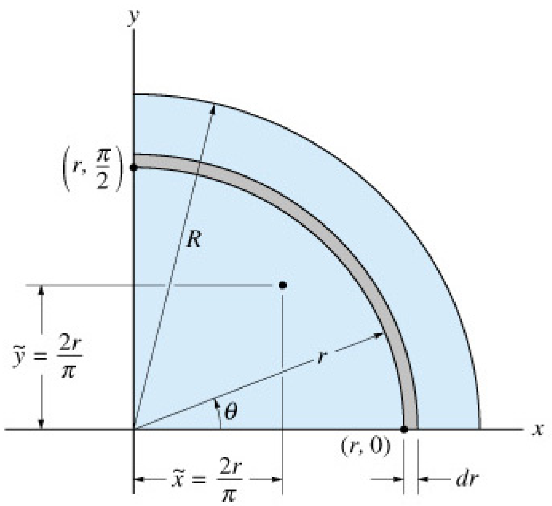
$$\bar{y} = \frac{\int \tilde{y}\,dA}{\int dA} \qquad dA = x\,dy$$

$$x = \left(b - \frac{b}{h} y\right)$$

$$\int \tilde{y}\, dA = \int_0^h y\, x\, dy = \int_0^h y\left(b - \frac{b}{h} y\right) dy = \frac{b}{h}\int_0^h y(h-y)\, dy$$

$$\frac{b}{h}\left(\frac{h y^2}{2} - \frac{y^3}{3}\right)\Big|_0^h = \frac{b}{h}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = \frac{b}{h} \times \frac{h^3}{6} = \frac{b h^2}{6}$$

$$\bar{y} = \frac{\frac{b h^2}{6}}{\frac{b h}{2}} = \frac{h}{3}$$



(b)

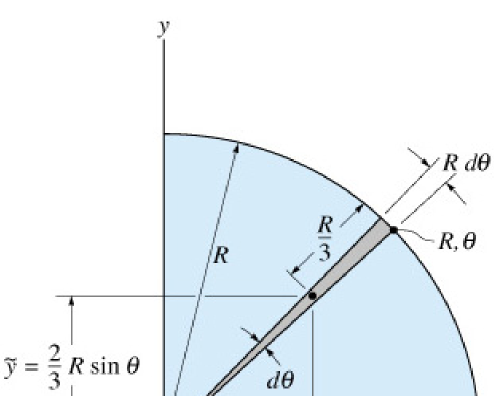
$$\bar{x} = \frac{\int \tilde{x}\, dA}{\int dA}$$

مرکز سطح ؟

$$dA = \frac{\pi r}{2} dr \qquad \tilde{x} = \frac{2r}{\pi} = \tilde{y}$$

$$\int \tilde{x}\, dA = \int_0^R \frac{2r}{\pi}\, \frac{\pi r}{2} dr = \int_0^R r^2 dr = \frac{r^3}{3}\Big|_0^R = \frac{R^3}{3}$$

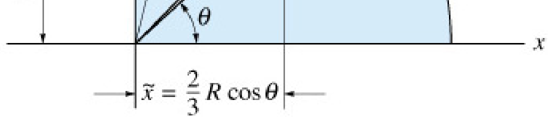
$$\bar{x} = \bar{y} = \frac{\frac{R^3}{3}}{\frac{\pi R^2}{4}} = \frac{4R}{3\pi}$$



$$dA = \frac{R^2 d\theta}{2} \qquad \tilde{x} = \frac{2}{3} R \cos\theta$$

$$\tilde{y} = \frac{2}{3} R \sin\theta$$

$$\int \tilde{x}\, dA = \int \frac{2}{3} R \cos\theta \times \frac{R^2}{2} d\theta = \frac{R^3}{3} \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta$$

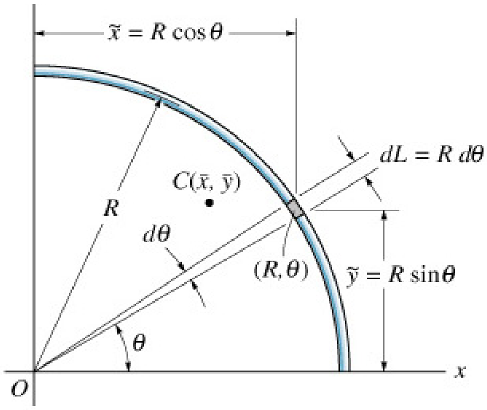


(a)

$$\bar{x} = \frac{\frac{\ldots}{3}}{\frac{\pi R^2}{4}} = \frac{4R}{3\pi} = \bar{y}$$

$$= \frac{R}{3}$$

مثال:



$$\bar{x} = \frac{\int \tilde{x}\, dL}{\int dL}$$

$dL = R\,d\theta$

$\tilde{x} = R\cos\theta$

$\tilde{y} = R\sin\theta$

$$\left.\begin{aligned} \int \tilde{x}\, dL &= \int_0^{\frac{\pi}{2}} R^2 \cos\theta\, d\theta = R^2 \sin\theta \Big|_0^{\frac{\pi}{2}} = R^2 \\ \int dL &= \int R\, d\theta = R\theta \Big|_0^{\frac{\pi}{2}} = \frac{\pi R}{2} \end{aligned}\right\} \quad \bar{x} = \frac{R^2}{\frac{\pi R}{2}} = \frac{2}{\pi} R = \bar{y}$$

$$\int \tilde{y}\, dL = \int R^2 \sin\theta\, d\theta = -R^2 \cos\theta \Big|_0^{\frac{\pi}{2}} = R^2$$